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IRREDUCIBLE DOMAINS IN BANACH SPACES

BY

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ABSTRACT

Every bounded domain in a complex Banach space E is biholomorphically equivalent to a finite product of irreducibles if and only if E does not contain c_0. A quantitative version of this holds if and only if E has finite cotype.

§1. Introduction

In [16, p. 17, problem II] W. Kaup posed the following problem: "For which complex Banach spaces E is every bounded domain $D \subset E$ biholomorphically equivalent to a direct product of irreducible complex Banach manifolds?"

In [16], Kaup gave a partial answer and in this article we give a complete solution. Our main results are the following two.

THEOREM 1.1. *Every bounded domain in the complex Banach space E is biholomorphically equivalent to a finite product of irreducible complex Banach manifolds if and only if E does not contain a subspace isomorphic to c_0.*

Of course, reflexive spaces (for example) do not contain c_0.

THEOREM 1.2. *The Banach space E with open unit ball B has finite cotype if and only if there exists a function $\phi : [1,\infty) \to [1,\infty)$ such that if D is a domain in E, $B \subset D \subset rB$ and D is biholomorphically equivalent to $D_1 \times D_2 \times \cdots \times D_n$ where each D_i is a complex Banach manifold of positive dimension, then $n \leqq \phi(r)$.*

This quantitative version of Theorem 1.1 applies, for example, in uniformly convex spaces, which always have finite cotype. One might refer to r as the *eccentricity* of the domain D. Clearly, we could assume only $r_1 B \subset D \subset r_2 B$, and have the same conclusion with $r = r_2/r_1$.

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Our methods include the use of invariant metrics from complex analysis and various concepts and results from Banach space theory. We discuss some of these and also prove some technical results for later use in sections 2 and 3. In section 4 we prove Theorem 1.1 and in section 5 we prove Theorem 1.2 together with some related results. In section 6 we give relations between the function ϕ and various functions which arise in the geometric theory of Banach spaces and we also give explicit estimates of ϕ for $L^p(\mu)$ spaces. Some of the estimates we require may be of independent interest.

We refer to [8, 12] for basic concepts in infinite dimensional holomorphy, to [12, 17, 18] for the theory of invariant metrics on complex manifolds and to [7, 19, 20, 25] for results on the geometry of Banach spaces.

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§2. Banach space theory

We first recall ([7, p. 44, theorem 6] and [19, p. 98, proposition 2.e.4]) a characterization of Banach spaces which contain c_0.

Proposition 2.1. *A Banach space E has a subspace isomorphic to c_0 if and only if there is a non-convergent series $\sum_{n=1}^{\infty} x_n$ in E such that, for some $c > 0$,*

$$\sup_N \left\| \sum_{n=1}^{N} \alpha_n x_n \right\| \leqq c \sup_n |\alpha_n|$$

holds for all sequences of scalars $(\alpha_n)_{n=1}^{\infty}$.

Next we construct a bounded domain in a Banach space E containing c_0 with certain properties which are used in section 4 to show that this domain is not biholomorphically equivalent to any finite product of irreducible domains. We remark that if E is separable then c_0 is complemented in E and if E is a dual space then l_∞ is contained and complemented in E and consequently our construction is trivially achieved in both of these cases.

Lemma 2.2. *Let $(E, \|\cdot\|)$ be a Banach space. If $F \subset E$ is a closed subspace and $\|\cdot\|_1$ is an equivalent norm on F, then $\|\cdot\|_1$ can be extended to an equivalent norm on E.*

Proof. Let C be the closed unit ball for the norm $\|\cdot\|_1$ on F and let $\bar{B}_F$, $\bar{B}_E$ be the closed unit balls for $\|\cdot\|$ on F and E, respectively. Then

$$m\bar{B}_F \subset C \subset M\bar{B}_F$$

for some $0 < m < M < \infty$ and the closed convex hull of $C \cup mB_E$ is the unit ball for the required norm on E. (We thank the referee for suggesting this generalization of our original lemma and its proof.)

REMARK 2.3. If a Banach space $(E, \|\cdot\|)$ contains a subspace isomorphic to c_0, then there is a $(1+\varepsilon)$-equivalent norm $\|\cdot\|_a$ on E (for any prescribed $\varepsilon > 0$) so that $(E, \|\cdot\|_a)$ has a subspace isometrically isomorphic to c_0. This follows from a theorem of James [7, p. 24] and Lemma 2.2.

LEMMA 2.4. *If c_0 is isomorphic to a subspace of E, then there exists an equivalent norm on $(E, \|\cdot\|_b)$, such that for each n there exists a Banach space F_n and $(E, \|\cdot\|_b)$ is isometrically isomorphic to $l_\infty^n \oplus_\infty F_n$ for all n. (l_∞^n denotes $\mathbf{C}^n$ endowed with the supremum norm and $\oplus_\infty$ signifies the maximum norm on $l_\infty^n \oplus F_n$.)*

PROOF. By Lemma 2.2, we may suppose c_0 is isometrically embedded in E. Let T denote the isometric embedding. Let J denote the canonical (and isometric) embedding of a Banach space in its second dual. The following diagram is commutative:

$$\begin{array}{ccc} c_0 & \xrightarrow{\quad T \quad} & E \\ J\downarrow & & \downarrow J \\ l_\infty & \xrightarrow{\quad \hat{T} \quad} & E'' \end{array}$$

where $\hat{T}$ denotes the second adjoint of T.

Now l_∞ is complemented by a norm one projection in E'' [7, p. 71] and we let P denote a norm one projection from E'' onto $T(l_\infty)$.

We renorm E'' by

$$\|x\|_{\hat{b}} = \sup(\|Px\|, \tfrac{1}{2}\|(I-P)x\|)$$

and we let $\|x\|_b = \|Jx\|_{\hat{b}}$ for every x in E. Since J and T are isometries, it is clear that $\|\cdot\|_b$ is equivalent to the original norm on E.

By using P and the canonical projection from l_∞ onto l_∞^n we see that there exists a bicontractive projection P_n from $(E'', \|\cdot\|_{\hat{b}})$ onto $\hat{T}(l_\infty^n)$. Hence $(E'', \|\cdot\|_{\hat{b}})$ is isometrically isomorphic to $l_\infty^n \times (I - P_n)E''$. Since $\hat{T}(l_\infty^n) \subset J(E)$ and $J(E)$ is a subspace of E'' we see that $(E, \|\cdot\|_b)$ is isometrically isomorphic to $l_\infty^n \times F_n$, where $F_n = (I - P_n)E'' \cap J(E)$. This completes the proof.

REMARK 2.5. Using Remark 2.3, we may suppose that

$$(1+\varepsilon)\|x\| \geqq \|x\|_b \geqq \tfrac{1}{3}\|x\| \qquad \text{for any prescribed } \varepsilon > 0.$$

§3. Invariant metrics on complex Banach manifolds

If D is a complex Banach manifold modelled on the Banach space $(E, \|\cdot\|)$, we let $T_x D$ denote the complex tangent space to D at x and let $T(D)$ denote the complex tangent manifold of D. Let $H(D_1, D_2)$ denote the space of all holomorphic mappings from the complex Banach manifold D_1 into the complex Banach manifold D_2 and let Δ denote the open unit disc in $\mathbf{C}$. The Kobayashi differential metric on D is the function $k_D : T(D) \to \mathbf{R}$ defined by

$$k_D(x, v) = \inf\{|\lambda| : \exists h \in H(\Delta, D), h(0) = x, \lambda h'(0) = v\}.$$

For each x in D, the mapping $v \mapsto k_D(x, v)$ defines a non-negative length function on $T_x D$ satisfying

$$k_D(x; \lambda v) = |\lambda| k_D(x, v)$$

but not necessarily satisfying the triangle inequality.

For our purposes the failure of the triangle inequality will not be important, partly because $k_D(x, \cdot)$ will be "equivalent" to a norm on E. We remark that the Carathéodory differential metric $c_D(x; v)$, which is defined dually to $k_D(x; v)$, does always give a seminorm on $T_x D$. By [9, theorem 2.5], if D is a convex domain in E, then $k_D(x; \cdot) = c_D(x; \cdot)$ is a seminorm.

The fundamental properties of k_D we will need are as follows.

PROPOSITION 3.1. (i) *If D_1, D_2 are complex Banach manifolds, $f \in H(D_1, D_2)$ and $x \in D_1$, then*

$$k_{D_2}(f(x); f'(x)v) \leqq k_{D_1}(x; v)$$

for all $v \in T_x(D_1)$. In particular if f is a biholomorphic mapping, then $f'(x)$ is an "isometry" with respect to $k_{D_1}(x; \cdot)$ and $k_{D_2}(f(x); \cdot)$.

(ii) *If $D_1, D_2, \ldots, D_n$ are complex Banach manifolds, $D = D_1 \times D_2 \times \cdots \times D_n$, $x = (x_1, x_2, \ldots, x_n) \in D$ and $v = (v_1, v_2, \ldots, v_n) \in T_x D$, then*

$$k_D(x; v) = \max_j k_{D_j}(x_j, v_j).$$

EXAMPLE 3.2. If follows from the Hahn–Banach theorem and the Schwarz lemma that if $D = rB$, where B is the unit ball of a Banach space, then

$$k_D(0; v) = \|v\|/r.$$

A complex Banach manifold (for instance, a domain in E) is called *irreducible* if it is not biholomorphic to a product of two such manifolds of positive dimension.

PROPOSITION 3.3. *Let D be a bounded* circular *domain in a Banach space E (i.e., $0 \in D$ and if $x \in D$ then $e^{i\theta}x \in D$ for all $\theta \in \mathbf{R}$) and let $f: D \to D_1 \times D_2$ be a biholomorphic mapping from D onto a product of two complex Banach manifolds. Then there are biholomorphic mappings $g_j: D_j \to \tilde{D}_j$ $(j = 1, 2)$ of D_j onto bounded circular domains $\tilde{D}_j$ in Banach spaces E_j such that*

$$F = (g_1, g_2) \circ f: D \to \tilde{D}_1 \times \tilde{D}_2$$

is linear.

PROOF. The existence of $\tilde{D}_j$ and g_j such that $F(0) = (0, 0)$ is shown by Braun, Kaup and Upmeier [4, p. 129]. But then F must be linear (see the proof on p. 76 of [12]).

THEOREM 3.4. *Suppose the unit ball B of a Banach space E is biholomorphic to a finite product of irreducible complex Banach manifolds. Then there are closed subspaces $E_1, E_2, \ldots, E_n$ of E, which are unique up to their order, satisfying*:

(i) $E = E_1 \oplus E_2 \oplus \cdots \oplus E_n$,

(ii) $B = (B \cap E_1) \times \cdots \times (B \cap E_n)$,

(iii) *for each j, $B \cap E_j$ is irreducible.*

Moreover,

(iv) *if B is biholomorphic to a product $D_1 \times D_2 \times \cdots \times D_m$ of irreducible complex Banach manifolds, then $m = n$ and D_k is biholomorphic to $B \cap E_{j_k}$ for some permutation $j_1, j_2, \ldots, j_n$ of $1, 2, \ldots, n$,*

(v) *if $F_1, F_2, \ldots, F_m$ are positive dimensional closed subspaces of E with the properties,*

(i)′ $E = F_1 \oplus F_2 \oplus \cdots \oplus F_m$,

(ii)′ $B = (B \cap F_1) \times (B \cap F_2) \times \cdots \times (B \cap F_m)$,

then $m \leqq n$ and each F_k is the sum of those of the E_j's which it contains.

PROOF. Suppose B is biholomorphic to a product $D_1 \times D_2 \times \cdots \times D_n$ of complex Banach manifolds of positive dimension. By Proposition 3.3 we can find Banach spaces $E_1, E_2, \ldots, E_n$, bounded domains $\tilde{D}_j \subset E_j$ which are biholomorphic to D_j $(j = 1, 2, \ldots, n)$ and a linear isomorphism

$$T: E \to E_1 \oplus E_2 \oplus \cdots \oplus E_n$$

such that

$$T(B) = \tilde{D}_1 \times \tilde{D}_2 \times \cdots \times \tilde{D}_n.$$

Replacing E_j by $T^{-1}(E_j) \subset E$, we may assume that T is the identity operator and that $E_1, E_2, \ldots, E_n$ are subspaces of E. This shows (i), (ii), (iii).

Now let $P_j: E \to E_j$ be the projection with kernel $E_1 \oplus \cdots \oplus E_{j-1} \oplus E_{j+1} \oplus \cdots \oplus E_n$ $(1 \leqq j \leqq n)$. We then have

$$\| x \| = \max(\| P_1 x \|, \ldots, \| P_n x \|)$$

for $x \in E$, which implies that each of the projections P_j is an M-projection (in the terminology of [2]) — i.e. $\| x \| = \max(\| P_j x \|, \| x - P_j x \|)$.

It is a fact that any two M-projections commute (see [2, p. 16] for a simple proof involving duality) and the rest of the theorem follows quite easily. In fact, if $Q_1, Q_2, \ldots, Q_m$ are any projections satisfying

$$\| x \| = \max(\| Q_1 x \|, \ldots, \| Q_m x \|)$$

then $P_j Q_k = Q_k P_j$ are projections and thus

$$E_j = Q_1 E_j \oplus \cdots \oplus Q_m E_j,$$

and

$$B \cap E_j = (B \cap Q_1 E_j) \times \cdots \times (B \cap Q_m E_j).$$

Now irreducibility of $B \cap E_j$ implies that $B \cap E_j = B \cap Q_{k_j} E_j$ for some k_j. Thus

$$E_j \subset F_{k_j}$$

and

$$E_j \cap F_k = 0 \qquad \text{if } k \neq k_j$$

(for $F_1, F_2, \ldots, F_m$ as in (v)). Similarly

$$\begin{aligned} F_k &= P_1 F_k \oplus \cdots \oplus P_n F_k \\ &= (E_1 \cap F_k) \oplus \cdots \oplus (E_n \cap F_k) \end{aligned}$$

which implies that F_k is the direct sum of the E_j's which it contains. Also $m \leqq n$ must hold. This is part (v) of the theorem.

If we now assume that $B \cap F_k$ is irreduible for all k, we find $m = n$ and $F_1, F_2, \ldots, F_n$ must be a reordering of $E_1, \ldots, E_n$, which is the uniqueness.

Finally (iv) follows by repeating the beginning of the proof and using the uniqueness.

§4. Proof of Theorem 1.1

Lemma 2.4 and Theorem 3.4 toegether imply that if c_0 is contained in E, then E contains a bounded domain (the unit ball for an equivalent norm on E) which is not biholomorphically equivalent to a finite product of irreducibles. Theorem 1.1 will follow from this and the following proposition.

PROPOSITION 4.1. *If c_0 is not contained in a Banach space E, then every bounded domain in E is biholomorphically equivalent to a finite product of irreducible complex Banach manifolds.*

PROOF. Suppose the conclusion is false and that the bounded domain D is not biholomorphically equivalent to any finite product of irreducible domains. Let B denote the open unit ball of $(E, \|\cdot\|)$. We may suppose without loss of generality that $B \subset D \subset rB$ for some $r > 1$. Hence, using Proposition 3.1(i) and Example 3.2, we have

$$r^{-1}\| v \| = k_{rB}(0, v) \leqq k_D(0, v) \leqq k_B(0, v) = \| v \|$$

for all $v \in T_0(D) = E$.

By hypothesis D is not irreducible and hence D is biholomorphically equivalent to a product of Banach manifolds $D_1 \times G_2$ one of which, G_2 say, is not biholomorphically equivalent to a finite product of irreducible Banach manifolds. By an obvious inductive argument we find two sequences of complex Banach manifolds $(D_i)_{i=1}^{\infty}$ and $(G_i)_{i=2}^{\infty}$ and two sequences of holomorphic mappings $(f_i)_{i=1}^{\infty}$ and $(g_i)_{i=2}^{\infty}$ such that

(a) $\dim(D_i) \geqq 1$,

(b) $f_i \in H(D, D_i)$ for all $i \geqq 1$ and $g_i \in H(D, G_i)$ for all $i \geqq 2$,

(c) $h_n = (f_1, \ldots, f_n, g_{n+1})$: $D \to D_1 \times \cdots \times D_n \times G_{n+1}$ is biholomorphic.

We suppose D_i (resp. G_i) is modelled on the Banach space E_i (resp. F_i) for all i. The linear mapping

$$h_n'(0): E \to E_1 \oplus E_2 \oplus \cdots \oplus E_n \oplus F_{n+1}$$

is an isomorphism and hence gives a direct sum decomposition of E. For each i choose $v_i \in E_i$ such that $k_{D_i}(f_i(0); v_i) = 1$ and let

$$u_i = h_n'(0)^{-1}(v_i).$$

For any sequence of scalars $(\alpha_i)_{i=1}^{\infty}$ we have for all n

$$\begin{aligned} \left\| \sum_{i=1}^{n} \alpha_i u_i \right\| &\leqq r k_D \left(0; \sum_{i=1}^{n} \alpha_i u_i\right) \\ &= r k_{D_1 \times D_2 \times \cdots \times D_n} \left(h_n(0), \sum_{i=1}^{n} \alpha_i v_i\right) \qquad (4.1) \\ &= r \sup_{i=1,\ldots,n} |\alpha_i|. \end{aligned}$$

We also have

$$\left\| \sum_n^m u_i \right\| \geqq k_D \left(0, \sum_{i=n}^m u_i\right) = \sup_{n \leqq i \leqq m} k_{D_i}(f_i(0), v_i) = 1. \tag{4.2}$$

By (4.1), (4.2) and Proposition 2.1 if follows that $c_0 \subset E$. This is a contradiction and completes the proof.

§5. Proof of Theorem 1.2

NOTATION. If D is a bounded domain in a Banach space we let $\pi(D)$ denote the supremum of all n for which D is biholomorphically equivalent to a product $D_1 \times D_2 \times \cdots \times D_n$ of complex Banach manifolds of positive dimension. Of course $\pi(D) = +\infty$ is possible in general.

We define $\pi L(D)$ analogously, but allowing only linear biholomorphic mappings from D onto a product of domains in Banach spaces.

DEFINITION 5.1. If E is a Banach space with open unit ball B, we let

$$\phi_E(r) = \sup\{\pi(D): B \subset D \subset rB\}.$$

If $\phi_E(1) = 1$, i.e. if B is irreducible, we let

$$\bar{\phi}_E = \sup\{r: \phi_E(r) = 1\} = \inf\{r: \phi_E(r) > 1\}.$$

We call ϕ_E the *irreducibility function* and $\bar{\phi}_E$ the *irreducibility radius* of E. Theorem 1.2 says that $\phi_E(r) < \infty$ for all r if and only if E has finite cotype. If D is a domain in E and $B \subset D \subset rB$ with $r < \bar{\phi}_E$, then D is irreducible and $\bar{\phi}_E$ is the largest constant with this property.

If E and F are isomorphic Banach spaces, then the *Banach–Mazur distance* between E and F, denoted $d(E, F)$, is defined as $\inf[\|T\| \, \|T^{-1}\|]$, where T ranges over all linear isomorphisms from E to F. (If E and F are not isomorphic then $d(E, F)$ is sometimes defined as $+\infty$.)

Our first result reduces the proof of Theorem 1.2 to a linear problem.

THEOREM 5.2. *If E is a Banach space (over* **C***) with unit ball B, then*
(i) $\phi_E(r) = \sup\{\pi L(D)$: *$B \subset D \subset rB$, D convex and circular*$\}$,
(ii) $\phi_E(r-0) = \sup\{n$: *$d(E, E_1 \oplus_\infty \cdots \oplus_\infty E_n) < r$ for some Banach spaces $E_1, E_2, \ldots, E_n$ of positive dimension*$\}$ $(r > 1)$.
(By $\phi_E(r-0)$ we mean the left hand limit at r or $\sup\{\phi_E(t) \mid t < r\}$.*)*

PROOF. (i) Let D be any domain in E with $B \subset D \subset rB$ and $f = (f_1, f_2, \ldots, f_n)$: $D \to D_1 \times D_2 \times \cdots \times D_n$ a biholomorphic map onto a product of manifolds. Then $T = f'(0)$ is an isometry from E with the Kobayashi differential metric $k_D(0;\cdot)$ to $E_1 \times E_2 \times \cdots \times E_n$ (where $E_j = T_{f_j(0)}(D_j)$) equipped with the corresponding Kobayashi metric

$$k_{D_1 \times \cdots \times D_n}(f(0); u) = \max k_{D_j}(f_j(0); u_j).$$

We now identify each E_j with the subspace $T^{-1}(E_j)$ of E and henceforth assume that T is the identity and $E_j \subset E$. Then we have $E = E_1 \oplus E_2 \oplus \cdots \oplus E_n$.

Now let $\tilde{D}$ be the convex hull of the indicatrix

$$\{v \in E \mid k_D(0; v) < 1\}.$$

Apply 3.1(a) to the inclusions $B \subset D \subset rB$ (and use 3.2) to get $\| v \| \geqq k_D(0; v) \geqq \| v \| / r$. Consequently $\tilde{D}$ is a balanced domain and $B \subset \tilde{D} \subset rB$.

The fact that $T =$ identity is an isometry may be restated as

$$\tilde{D} = (\tilde{D} \cap E_1) \times (\tilde{D} \cap E_2) \times \cdots \times (\tilde{D} \cap E_n).$$

Thus $\phi_E(r) \leqq \sup\{\pi L(D) \mid B \subset D \subset rB,\ D \text{ convex circular}\}$.

The reverse inequality is obvious and we have shown (i).

Part (ii) follows immediately from (i) and the observtion that a convex circular bounded domain in E is the unit ball for an equivalent norm on E.

NOTATION. If E is a Banach space let

$$C_E(n) = \inf\{d(F, l_\infty^n) \mid F \subset E \text{ is an } n\text{-dimensional subspace}\}.$$

Our next result shows that $\phi_E(r)$ and $C_E(n)$ are almost inverse to each other.

PROPOSITION 5.3. *For any Banach space E, any $r \geqq 1$ and any $n \geqq 1$ ($n \leqq$ dimension of E),*

(i) *if $\phi_E(r) \geqq n$ then $C_E(n) \leqq r$,*

(ii) *$\phi_E(2C_E(n) + 1 + \varepsilon) \geqq n + 1$ (for any $\varepsilon > 0$).*

PROOF. (i) Using 5.2(i), if $\phi_E(r) \geqq n$ we can find a convex circular domain $D \subset E$ with $B \subset D \subset rB$ and subspaces $E_1, E_2, \ldots, E_n \subset E$ with $E = E_1 \oplus E_2 \oplus \cdots \oplus E_n$ and $D = (D \cap E_1) \times (D \cap E_2) \times \cdots \times (D \cap E_n)$. Let $\| \cdot \|_D$ denote the norm on E which has D as its unit ball. Choose $v_j \in E_j$ with $\| v_j \|_D = 1$. Then

$$\left\| \sum_{j=1}^{n} \alpha_j v_j \right\|_D = \max_j \| \alpha_j v_j \|_D = \max_j | \alpha_j |$$

for any scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, which shows that the linear span F of $v_1, v_2, \ldots, v_n$

is isomorphic to l_∞^n. In fact, F with the norm $\|\cdot\|_D$ is isometric to l_∞^n and thus $d(F, l_\infty^n) \leqq r$ when we use the original norm. Hence $C_E(n) \leqq r$ and the proof of (i) is complete.

(ii) Given n, by the definition of $C_E(n)$, we may find a subspace $F \subset E$ and an isomorphism $T: F \to l_\infty^n$ satisfying $\|T\| \leqq 1$, $\|T^{-1}\| \leqq C_E(n) + \varepsilon$. By the Hahn–Banach theorem, there exists a projection $P: E \to F$ with range F and $\|P\| \leqq C_E(n) + \varepsilon$ (see [7, p. 71] for example). Now let

$$D_1 = T^{-1}(B_{l_\infty^n}) \subset F, \quad D_2 = (C_E(n) + 1 + \varepsilon)B \cap \ker(P) \quad \text{and} \quad D = D_1 \times D_2.$$

Then

$$B_E \subset D \subset (2C_E(n) + 1 + 2\varepsilon)B_E$$

and D has $n + 1$ factors since D_1 is a copy of an n-dimensional polydisc. This completes the proof of (ii).

We can now deduce the following, more precise version of Theorem 1.2.

THEOREM 5.4. *Let E be an (infinite dimensional) Banach space. Then the following are equivalent properties for E:*

(i) $\phi_E(r) < \infty$ *for all* $r \geqq 1$,

(ii) $\sup_n C_E(n) = \infty$,

(iii) $\sup_n C_E(n) > 1$,

(iv) $\phi_E(3 + \varepsilon) < \infty$ *for some* $\varepsilon > 0$,

(v) *E has some finite cotype.*

PROOF. The equivalence of (i) and (ii) clearly follows from Proposition 5.3. An outline of a proof that (ii) and (iii) are equivalent is given by Figiel [11, p. 203]. The original proof of this seems to be unpublished. It can also be proved by using James' theorem on c_0 [7, p. 241] and the theory of ultraproducts of Banach spaces.

Clearly (i) implies (iv) and (iv) implies (iii) by Proposition 5.3(ii).

The fact that (ii) (or (iii)) is equivalent to (v) is a deep result of Maurey and Pisier [21].

REMARKS 5.5. We will give the definition of cotype in §6 and show that $\phi_E(r)$ grows like a power of r, the exponent being related to the cotype of E.

In fact we can now give the weaker result that $\phi_E(r) \leqq cr^q$ for some $c > 0$ and some $q > 0$, assuming $\phi_E(r) < \infty$ for all r. This follows from Figiel's proof of the equivalence of (ii) and (iii). He actually shows that (iii) is equivalent to

(ii)′ $C_E(n) > cn^\varepsilon$ for some $c > 0$, $\varepsilon > 0$, all n.

From (ii)′ and Proposition 5.3(i) it follows that

$$\phi_E(cn^\varepsilon) < n.$$

Hence $\phi_E(r) < (r/c)^{1/\varepsilon} + 1$.

§6. Estimates and examples

In this section we make more precise the relationship between cotype of a Banach space E and the irreducibility function $\phi_E(r)$ introduced in Section 5. This allows us to calculate $\phi_E(r)$ exactly in the case $E = L^p(\mu)$, $p \geqq 2$ (μ any measure). However the best possible constants obtained by the same method in the case $p < 2$ do not give $\phi_E(r)$ exactly. This suggests using concepts which make use of complex scalars such as a certain absolutely summing norm of the identity on E or a complex version of cotype. The theory of complex uniform convexity [6] does not seem to have been developed sufficiently for us to use it successfully, although it is known to imply finite cotype.

DEFINITION 6.1. Let E be a Banach space and let $r_n(t)$ denote the Rademacher functions on $[0,1]$, $r_n(t) = \operatorname{sgn}\{\sin(2^n \pi t)\}$. E is said to have *cotype q* ($q \geqq 2$) if there is a constant $c < \infty$ so that, for every finite set of vectors $\{x_j\}_{j=1}^n$ in E, we have

$$\sum_{j=1}^n \|x_j\|^q \leqq c^q \int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\|^q dt. \tag{6.1}$$

The infimum of all constants c satisfying (6.1) for a fixed n is denoted by $C_{q,n}(E)$. We let $C_q(E) = \sup_n C_{q,n}(E)$ and call $C_q(E)$ the *q-cotype constant* of E. A Banach space is said to have *finite cotype* if it has cotype q for some $2 \leqq q < \infty$.

PROPOSITION 6.2. *For any Banach space E, the following inequalities hold:*

$$n \leqq C_{q,n}(E)^q C_E(n)^q, \tag{6.2}$$

$$\phi_E(r) \leqq C_q(E)^q r^q, \tag{6.3}$$

$$\bar{\phi}_E \geqq 2^{1/q} C_{q,2}(E)^{-q}. \tag{6.4}$$

PROOF. To prove (6.2) fix an $\varepsilon > 0$ and choose a finite dimensional subspace $F \subset E$ and an isomorphism $T: F \to l_\infty^n$ satisfying $\|T\| \leqq 1$, $\|T^{-1}\| \leqq C_E(n) + \varepsilon$. Then choose $x_1, x_2, \ldots, x_n \in E$ such that $T(x_j) = e_j =$ the jth standard basis vector e_j in l_∞^n. Then for $t \in [0,1]$,

$$\begin{aligned}\left\|\sum_{j=1}^{n} r_j(t)x_j\right\| &= \left\|T^{-1}\left(\sum_{j=1}^{n} r_j(t)e_j\right)\right\| \\ &\leqq \|T^{-1}\| \left\|\sum_{j=1}^{n} r_j(t)e_j\right\|_\infty \\ &= \|T^{-1}\| \\ &\leqq C_E(n)+\varepsilon.\end{aligned}$$

On the other hand,

$$\|x_j\| \geqq \|Tx_j\| = \|e_j\|_\infty = 1.$$

Combining these two inequalities with (6.1) and letting $\varepsilon \to 0$ yields (6.2). Proposition 5.3(i) now yields (6.3) and (6.4).

Recall that l_q^n denotes $\mathbf{C}^n$ with the norm

$$\|z\|_q = \left(\sum_{j=1}^{n} |z_j|^q\right)^{1/q}.$$

We say that a Banach space E contains l_q^n's almost isometrically if for each $n \geqq 1$ and each $\varepsilon > 0$ we can find an n-dimensional subspace $F \subset E$ with $d(F, l_q^n) < 1+\varepsilon$.

We will use the following deep result due to Maurey and Pisier [21].

THEOREM 6.3. *If E is an infinite dimensional Banach space, then*

$$\sup\{q : E \text{ contains } l_q^n\text{'s almost isometrically}\} = \inf\{q : E \text{ has cotype } q\}$$

(*where the infimum is taken to mean* $+\infty$ *if E does not have finite cotype*). *Moreover the above supremum is attained. We denote it by $q(E)$.*

Notice that Dvoretzky's theorem ([10], see [26] for the complex version) implies that the above supremum is always at least 2.

PROPOSITION 6.4. (i) *If $2 \leqq q_1 \leqq q_2 \leqq \infty$ then $d(l_{q_1}^n, l_{q_2}^n) = n^{1/q_1 - 1/q_2}$.*

(ii) *If $1 \leqq p \leqq 2 \leqq q \leqq \infty$ then $d(l_p^n, l_q^n) \leqq \max(n^{1/p-1/2}, n^{1/2-1/q})$ (for complex scalars).*

PROOF. For $2 \leqq q_1 \leqq q_2 \leqq \infty$ this is due to Gurarii, Kadec and Macaev [13] for real scalars and their proof also works for complex scalars. For $1 \leqq p \leqq 2 \leqq q \leqq \infty$, the proof they give for real scalars for n a power of 2 works for all n in the complex case since, for each n, there is an $n \times n$ unitary matrix with all its entries of modulus $1/\sqrt{n}$.

THEOREM 6.5. *If E is an infinite dimensional Banach space with finite cotype and $q = q(E)$ is as in Theorem 6.3, then the irreducibility function $\phi_E(r)$ satisfies*

$$\left(\frac{r-1}{2}\right)^q \leqq \phi_E(r) \leqq A(\varepsilon, E)r^{q+\varepsilon} \tag{6.5}$$

for all $\varepsilon > 0$ ($A(\varepsilon, E) = C_{q+\varepsilon}(E)^{q+\varepsilon} < \infty$ is a constant depending on E and ε).

PROOF. We have already observed (6.3) that the right hand inequality in (6.5) follows from the definition of the cotype constant $C_{q+\varepsilon}(E)$ and Proposition 5.3(i).

To prove the other inequality observe that Theorem 6.3 implies that E contains an n-dimensional subspace F with $d(F, l_q^n) < 1 + \varepsilon$. By Proposition 6.4(i) it follows that

$$d(F, l_\infty^n) < (1 + \varepsilon)n^{1/q}.$$

Hence $C_E(n) < (1 + \varepsilon)n^{1/q}$ and applying Proposition 5.3(ii) we obtain

$$\phi_E(2n^{1/q} + 1 + \varepsilon) \geqq n + 1.$$

Solving for n in terms of $r = 2n^{1/q} + 1 + \varepsilon$ yields the desired inequality.

In fact we obtain the result that the smallest integer greater than $((r-1)/2)^q$ is at most $\phi_E(r)$. This is already implied by (6.5) since $\phi_E(r)$ has integer values.

REMARK 6.6. We do not know whether it is possible to take $\varepsilon = 0$ in (6.5). A more interesting question might be whether the factor 2^{-q} on the left side of (6.5) can be removed. This would follow if the factor 2 could be removed in Proposition 5.3(ii).

Another question along these lines which we have been unable to answer is whether $\phi_E(r) \geqq [r^2]$ holds for all infinite dimensional Banach spaces E. Equality $\phi_E(r) = [r^2]$ holds when E is a Hilbert space (see Kaup [16] or Examples 6.10 below). Also, by (6.5) $\phi_E(r) \geqq ((r-1)/2)^2$ holds for all infinite dimensional Banach spaces E.

An even more ambitious question would be whether $\phi_E(r) \geqq [r^2]$ for $r^2 \leqq \dim E$, and E any (complex) Banach space. This seems likely to have a negative answer because of an example of a real two dimensional Banach space H with Banach–Mazur distance $\frac{3}{2}$ from the real space l_∞^2 [1].

DEFINITION 6.7. Let T^n denote the n-dimensional torus $\{(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n})$: $0 \leqq \theta_j \leqq 2\pi\}$ equipped with the probability measure

$$d\boldsymbol{\theta} = (2\pi)^{-n} d\theta_1 d\theta_2 \cdots d\theta_n.$$

For any complex Banach space E and $q \geqq 2$, $n \geqq 1$ we define $B_{q,n}(E)$ to be the least constant B satisfying

$$\left(\sum_{j=1}^{n} \|x_j\|^q\right)^{1/q} \leqq B\left(\int_{T^n} \left\|\sum_{j=1}^{n} e^{i\theta_j}x_j\right\|^q d\theta\right)^{1/q}$$

for arbitrary $x_1, x_2, \ldots, x_n \in E$.

We define the *complex q-cotype* constant of E to be

$$B_q(E) = \sup_n B_{q,n}(E).$$

DEFINITION 6.8. Let E be a complex Banach space, $q \geqq 2$, $n \geqq 1$. We define $A_{q,n}(E)$ to be the least constant A satisfying

$$\left(\sum_{j=1}^{n} \|x_j\|^q\right)^{1/q} \leqq A \sup\left\{\sum_{j=1}^{n} |x'(x_j)| : x' \in E', \|x'\| \leqq 1\right\}$$

for arbitrary $x_1, x_2, \ldots, x_n \in E$. The related constant

$$A_q(E) = \pi_{q,1}(E) = \sup_n A_{q,n}(E)$$

is the $(q, 1)$ absolutely summing norm of the identity on E.

$A_q(E)$ is finite if and only if every weakly absolutely summable series in E is q-absolutely summable. Moreover, if $q(E)$ is as in Theorem 6.3, it is known that $q(E) = \inf\{q : \pi_{q,1}(E) < \infty\}$.

Since

$$\sup\left\{\sum_{j=1}^{n} |x'(x_j)| : x' \in E', \|x'\| \leqq 1\right\} = \sup_{T^n} \left\|\sum_{j=1}^{n} e^{i\theta_j}x_j\right\|$$

it follows easily that

$$1 \leqq A_{q,n}(E) \leqq B_{q,n}(E) \leqq C_{q,n}(E), \tag{6.6}$$

$$1 \leqq \pi_{q,1}(E) = A_q(E) \leqq B_q(E) \leqq C_q(E). \tag{6.7}$$

As in Proposition 6.2 we can show the following estimates, which are sharper than (6.2), (6.3) and (6.4) in some instances.

PROPOSITION 6.9. *For any complex Banach space E, we have the following*:

$$n \leqq A_{q,n}(E)^q C_E(n)^q,$$

$$\phi_E(r) \leqq \pi_{q,1}(E)^q r^q,$$

$$\bar{\phi}_E \geqq 2^{1/q} A_{q,2}(E)^{-q}.$$

EXAMPLES 6.10.

(a) If $2 \leqq p < \infty$ and $E = L^p(X, \mu)$ is infinite dimensional then

$$\phi_E(r) = [r^p], \tag{6.8}$$

$$\bar{\phi}_E = 2^{1/p}. \tag{6.9}$$

PROOF. Clearly (6.9) follows from (6.8). We first show that $\phi_E(r) \leqq r^p$ (which immediately gives $\phi_E(r) \leqq [r^p]$ since $\phi_E(r)$ is integer valued). The inequality $\phi_E(r) \leqq r^p$ follows from inequality (6.3) and the fact that L^p has p-cotype constant 1 ($p \geqq 2$). This latter fact is well-known to follow from a trivial case of the Khintchine inequalities

$$\left(\int_0^1 \left| \sum_{j=1}^n r_j(t) a_j \right|^p dt \right)^{1/p} \geqq \left(\sum_{j=1}^n |a_j|^2 \right)^{1/2}$$

(valid for any scalars $a_1, a_2, \dots, a_n \in \mathbf{C}$ and any $p \geqq 2$) and an argument due to Orlicz [22].

To show $\phi_E(r) \geqq [r^p]$ we produce suitable examples of product domains in $L^p(X, \mu)$. Since $L^p(X, \mu)$ is infinite dimensional we can find a partition $X = X_1 \cup X_2 \cup \cdots \cup X_n$ of X into disjoint measurable sets such that $L^p(X_j, \mu \mid X_j)$ is non-zero for each j. The domain

$$D = \{ f \in L^p(X, \mu) \mid \| f \mid X_j \|_p \leqq 1 \text{ for } 1 \leqq j \leqq n \}$$

is clearly a product of n unit balls and has eccentricity $n^{1/p}$, i.e. $B \subset D \subset n^{1/p} B$ where B is the unit ball of $L^p(X, \mu)$. Thus $\phi_{L^p}(n^{1/p}) \geqq n$ which completes the proof of (6.8).

(b) If $E = L^p(X, \mu)$ is infinite dimensional and $1 \leqq p < 2$, we have the following results, which are less precise than the results for $p \geqq 2$:

$$\phi_E(r) \leqq \beta_p^2 r^2 \tag{6.10}$$

where $\beta_p = \Gamma((p+2)/2)^{-1/p}$,

$$\phi_E(r) > [(r-2)^2] + 1, \tag{6.11}$$

$$2^{1/p} \geqq \bar{\phi}_E \geqq \| 1 + e^{i\theta} \|_{L^p[0,2\pi]} = \left(\frac{\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}\, \Gamma\left(\frac{p+2}{2}\right)} \right)^{1/p}. \tag{6.12}$$

(For $p = 1$, (6.10) reads $\phi_E(r) \leqq (4/\pi) r^2$ and (6.12) reads $2 \geqq \bar{\phi}_{L^1} \geqq 4/\pi \simeq 1.273$).

PROOF. The inequality (6.10) follows from (6.3) and the fact that $E =$

$L^p(X,\mu)$ has a complex 2-cotype constant $C_2(E)=\beta_p$ with β_p as above. This follows from an argument due to Orlicz [22] similar to the one given above and the fact that β_p is the best constant in the complex version of the Khintchine inequality:

$$\beta_p\left(\frac{1}{(2\pi)^n}\int_0^{2\pi}\int_0^{2\pi}\cdots\int_0^{2\pi}\left|\sum_{j=1}^n a_je^{i\theta_j}\right|^p d\theta_1 d\theta_2\cdots d\theta_n\right)^{1/p}\geqq\left(\sum_{j=1}^n |a_j|^2\right)^{1/2}.$$

This latter fact is due to Sawa [23, 24] who used methods based on Haagerup [14]. Haagerup [14] determined the best constants for Khintchine inequalities involving Rademacher functions, improving on the work of Szarek [27].

To prove (6.11) we observe first that $E=L^p(X,\mu)$ contains a finite dimensional subspace E which is isometrically isomorphic to l_p^n and norm one complemented in E (for any $n\geqq 1$). To see this choose disjoint measurable subsets $X_1,X_2,\ldots,X_n\subset X$ satisfying $0<\mu(X_j)<\infty$. Take F to be the linear span of the characteristic functions χ_{X_j} $(1\leqq j\leqq n)$ and the norm one projection $P\colon E\to F$ to be

$$Pf=\sum_{j=1}^n\left(\frac{1}{\mu(X_j)}\int_{X_j} f\,d\mu\right)\chi_{X_j}.$$

Now Proposition 6.4(ii) implies $d(F,l_\infty^n)\leqq\sqrt{n}$ and by finite dimensionality we can therefore find a linear isomorphism $T\colon F\to l_\infty^n$ with $\|T\|\leqq 1$, $\|T^{-1}\|\leqq\sqrt{n}$. Now let $D_1=T^{-1}(B_{l_\infty^n})\subset F$, $D_2=2B_E\cap\ker(P)$, $D=D_1\times D_2$ where B_E denotes the unit ball of E. Observe that

$$B_E\subset D\subset(\sqrt{n}+2)B_E$$

and D has $n+1$ factors. Hence $\phi_E(\sqrt{n}+2)\geqq n+1$, which implies (6.11).

The first inequality in (6.12) is easily verified by constructing an example of a domain exactly as was done in showing $\phi_{L^p}(r)\geqq[r^p]$ for $p\geqq 2$. The other part of (6.12) follows from the fact that (for $1\leqq p\leqq 2$)

$$A_{2,2}(L_p(\mu,X))=B_{2,2}(L_p(\mu,x))=\sqrt{2}/\|1+e^{i\theta}\|_{L^p[0,2\pi]} \tag{6.13}$$

and Proposition 6.9.

Although facts similar to (6.13) can be found in the literature, we have not found the precise result. Formula (6.13) will be proved in Proposition 6.13 using the following two lemmas.

LEMMA 6.11. (i) *If* $0<r<1$, $-1\leqq t\leqq 1$, $0\leqq\alpha\leqq 1$, *then*

$$(1+\alpha t)^r+(1-\alpha t)^r\geqq(1+t)^r+(1-t)^r.$$

(ii) *If* $p \geqq 1,\ -1 \leqq t \leqq 1,\ z \in \mathbf{C}$, *then*

$$|1+tz|^p + |1-tz|^p \leqq |1+z|^p + |1-z|^p.$$

The proof is left to the reader.

LEMMA 6.12. *If* $a,\ b \in \mathbf{C}$ *and* $0 < p \leqq 2$, *then*

$$(|a|^2 + |b|^2)^{1/2} \leqq K_p \| a + be^{i\theta} \|_{L^p[0,2\pi]}$$

with $K_p = \sqrt{2}/\|1 + e^{i\theta}\|_{L^p[0,2\pi]}$. *Equality holds if* $|a| = |b|$.

PROOF. For the case $p = 1$ this lemma is given by Bennett [3]. It is easy to verify that equality holds if $|a| = |b|$. To prove the inequality it is sufficient to consider the case $a > 1 = b$. Then

$$\begin{aligned}\| a + e^{i\theta} \|_{L^p[0,2\pi]} &= \int_0^{2\pi} |a + e^{i\theta}|^p d\theta/2\pi \\ &= \tfrac{1}{2} \int_0^{2\pi} |a + e^{i\theta}|^p + |a - e^{i\theta}|^p d\theta/2\pi \\ &= \tfrac{1}{2}(a^2+1)^{p/2} \int_0^{2\pi} (1 + \alpha \cos\theta)^{p/2} + (1 - \alpha\cos\theta)^{p/2} d\theta/2\pi\end{aligned}$$

with $\alpha = 2a/(a^2+1) < 1$. Apply Lemma 6.11(i) with $t = \cos\theta$, $r = p/2$ to complete the proof.

PROPOSITION 6.13. *If* $f,\ g \in L_p(\mu, X)$, $1 \leqq p \leqq 2$, *then*

$$(\|f\|_p^2 + \|g\|_p^2)^{1/2} \leqq K_p \left(\int_0^{2\pi} \|f + e^{i\theta} g\|_p^p d\theta/2\pi \right)^{1/p}$$

with $K_p = \sqrt{2}/\|1 + e^{i\theta}\|_{L_p[0,2\pi]}$. *Hence* $B_{2,2}(L_p) = K_p$. *If* $L_p(\mu, x)$ *is infinite dimensional then* $A_{2,2}(L_p(\mu, X)) = B_{2,2}(L_p(\mu, X))$.

PROOF. The deduction of the inequality from Lemma 6.12 is the argument due to Orlicz [22].

Equality holds when $f = g =$ the characteristic function of a subset of X of finite measure. Hence $B_{2,2}(L_p) = K_p$.

To show $A_{2,2} = B_{2,2}$ in the infinite dimensional case we need only consider the case where $X = [0, 2\pi]$, $d\mu = dx/2\pi$ since $L_p[0,2\pi]$ is finitely representable in any infinite dimensional $L_p(\mu, X)$. Take $f \equiv 1$, $g = e^{ix}$. Then

$$\|f + e^{i\theta} g\|_p = \|f + g\|_p = \int_0^{2\pi} \|f + e^{i\theta} g\|_p d\theta/2\pi = \sqrt{2}/K_p$$

which implies that $A_{2,2}(L_p[0,2\pi]) \geqq K_p = B_{2,2}$. In general we have $A_{2,2} \leqq B_{2,2}$ (see (6.6)). Thus $A_{2,2} = B_{2,2}$ in this case.

Since it can now be done with very little effort, we show the following:

PROPOSITION 6.14. *If $E = L_p(\mu, X)$ is infinite dimensional and $1 \leqq p \leqq 2$, then $C_E(2) = \|1 + e^{i\theta}\|_p$. (For $p = 1$, this says $C_{L_1}(2) = 4/\pi$ in constrast to the case of real scalars where $C_{L_1}(2) = 1$.)*

PROOF. By finite representability of $L_p[0,2\pi]$ in $L_p(\mu, X)$, it is sufficient to consider the case $E = L_p[0,2\pi]$. Let f be the constant function 1 and $g(x) = e^{ix}$. Define T from the linear span of f and g to l_∞^2 by

$$T(af + bg) = (a, b).$$

Lemma 6.11(ii) can be used to check that $\|T\| \|T^{-1}\| \leqq \|1 + e^{ix}\|_p$.

The fact that $C_E(2) \geqq \|1 + e^{ix}\|_p$ follows from Proposition 6.13 and Proposition 6.9.

Our final result is an estimate on $\phi_E(r)$ in terms of the modulus of uniform convexity of E, $\delta_E(\varepsilon)$.

PROPOSITION 6.15. *If E is a uniformly convex space with modulus of convexity*

$$\delta_E(\varepsilon) = \inf\{1 - \|x + y\|/2 \colon x, y \in E, \|x\| = \|y\| = 1, \|x - y\| = \varepsilon\}$$

then $\phi_E(r) \leqq 1/\delta_E(2/r)$.

PROOF. By (for instance) Theorem 5.2(i), if $\phi_E(r) = n$, then we can find unit vectors $x_1, x_2, \ldots, x_n \in E$ such that $\|e^{i\theta_1}x_1 + e^{i\theta_2}x_2 + \cdots + e^{i\theta_n}x_n\| \leqq r$ for all $\theta_1, \theta_2, \ldots, \theta_n \in \mathbf{R}$. Consequently

$$\left\| \varepsilon_1 \frac{2x_1}{r} + \varepsilon_2 \frac{2x_2}{r} + \cdots + \varepsilon_n \frac{2x_n}{r} \right\| \leqq 2$$

for all choices of $\varepsilon_j = \pm 1$. Now by [7, p. 129] or [20, p. 70], this implies

$$\sum_{j=1}^{n} \delta_E\left(\left\|\frac{2x_j}{r}\right\|\right) = n\delta_E\left(\frac{2}{r}\right) \leqq 1$$

which is the desired result.

REMARKS 6.16. It follows that uniformly convex spaces E (i.e. $\delta_E(\varepsilon) > 0$ for all $\varepsilon > 0$) must have finite cotype. However L_1 has cotype 2 although it is not uniformly convex.

It is known that $L_p(\mu, X)$ is uniformly convex for $1 < p < \infty$. For $p > 1$, the modulus of convexity δ_{L_p} has been calculated (see Hanner [15], Clarkson [5]). The explicit expression for δ_{L_p} with $p \geqq 2$ and Proposition 6.15 yield

$$\phi_{L_p}(r) \leqq \{1 - (1 - r^{-p})^{1/p}\}^{-1} \qquad (p \geqq 2)$$

which is less precise than the inequality obtained by using the p-cotype constant of L_p in Example 6.10(a).

It seems natural to use a modulus of complex uniform convexity (as defined by [6]) to obtain a sharper result, but precise estimates of the kind used in the proof of Proposition 6.15 do not seem to be known.

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